# A GENERALIZATION OF THE RIESZ THEORY OF COMPLETELY CONTINUOUS TRANSFORMATIONS(1)

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The classical theory, due to F. Riesz, of completely continuous transformations deals with a family $T_c = I - cK$ of linear continuous transformations of a Banach space $\mathfrak{X}$ into itself, where $I$ is the identity and $K$ is completely continuous. The transformation $T_c$ is then one-to-one, except for the "proper" values of $c$. In this paper we consider cases where the one-to-oneness no longer holds. The identity transformation $I$ is replaced by a transformation $E$ which maps $\mathfrak{X}$ onto the whole of another Banach space $\mathfrak{Y}$, but *not* one-to-one. Then when $K$ is completely continuous, the transformation $T_c = E - cK$ maps $\mathfrak{X}$ onto $\mathfrak{Y}$ except for a countable set of proper values $c_i$ which have no finite accumulation point. When $\mathfrak{W}_1 = T_c\mathfrak{X} \neq \mathfrak{Y}$, $c$ is said to be a *proper value* for $K$. We can no longer iterate the transformation $T_c$, but we can apply $T_c$ to $E^{-1}T_c\mathfrak{X}$ and obtain a linear closed space $\mathfrak{W}_2 \subset \mathfrak{W}_1$. After a finite number of repetitions of this process, we find a space $\mathfrak{W}_\nu = \mathfrak{W}_{\nu+1}$. Then $T_c$ transforms $E^{-1}\mathfrak{W}_\nu$ onto $\mathfrak{W}_\nu$, which takes the place of the invariant subspace of the Riesz theory.

In the one-to-one case, there is associated with the transformation $I - cK$ a resolvent $R_c$, which has a pole of order $\nu$ at a proper value $c_0$, and the Laurent expansion of $R_c$ about $c_0$ leads to a decomposition of $K$ into two mutually orthogonal parts. In the many-to-one case, the adjoint transformation $T_c^*$ is one-to-one except when $c$ is a proper value, but the range of $T_c^*$ varies with $c$, so that it is not possible to define the order of a pole of the resolvent in the usual way. However a substitute definition has been found, as well as a decomposition $K = G + H$, where $G$ has only one proper value $c_0$, and $H$ has only the remaining proper values of $K$, and $H$ is semi-orthogonal to $G$ in a suitable sense.

1. **Notations and definitions.** We shall let $\mathfrak{X}$ and $\mathfrak{Y}$ denote Banach spaces of infinitely many dimensions, with complex scalars. The adjoint spaces of $\mathfrak{X}$ and $\mathfrak{Y}$ will be denoted by $\mathfrak{X}^*$ and $\mathfrak{Y}^*$ respectively. Two subsets $\mathfrak{X}_0$ of $\mathfrak{X}$ and $\mathfrak{X}_0^*$ of $\mathfrak{X}^*$ are said to be *orthogonal* in case $x^*(x) = 0$ for all $x$ in $\mathfrak{X}_0$ and $x^*$ in $\mathfrak{X}_0^*$. The *orthogonal complement* of a subset $\mathfrak{X}_0$, denoted by $O\mathfrak{X}_0$, is the maximal set $\mathfrak{X}_0^*$ orthogonal to $\mathfrak{X}_0$. It is easily seen that $O\mathfrak{X}_0$ always exists (possibly void) and is a linear closed set. A corresponding statement holds for $O\mathfrak{X}_0^*$.

A linear transformation $A$ mapping $\mathfrak{X}$ into $\mathfrak{Y}$ has an *adjoint* $A^*$ mapping $\mathfrak{Y}^*$ into $\mathfrak{X}^*$, defined by $x^* = A^*y^*$ where $x^*(x) = y^*Ax$ for each $x$. We shall

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deal only with continuous linear transformations $A$. Such a transformation $A$ is said to be *completely continuous* in case the transform $A(\mathfrak{X}_0)$ of every bounded set $\mathfrak{X}_0$ is compact (in the sense that every sequence $(y_n)$ chosen from $A(\mathfrak{X}_0)$ has a convergent subsequence).

We shall use the letter $E$ to denote a linear continuous transformation mapping $\mathfrak{X}$ onto the whole space $\mathfrak{Y}$, and shall use $K$, $G$, and $H$ to denote completely continuous linear transformations of $\mathfrak{X}$ into $\mathfrak{Y}$. We also set $T_c = E - cK$, where $c$ is a complex parameter. When it is convenient to set $c=1$, we shall write $T$ for $T_c$.

A *projection* $P$ is a linear continuous transformation of $\mathfrak{X}$ onto a closed linear subset $\mathfrak{X}_0$, which equals the identity on $\mathfrak{X}_0$, i.e., $P^2 = P$.

For a given linear continuous transformation $A$ of $\mathfrak{X}$ into $\mathfrak{Y}$ we shall set

$$\beta(A, y) = \text{g.l.b. } [\|x\| \mid Ax = y]$$

for each $y$, and

$$I(A) = \text{l.u.b.} [\beta(A, y) \mid y \in A\mathfrak{X}, \|y\| = 1].$$

When the transformation $A$ is one-to-one, obviously $I(A) = \|A^{-1}\|$. (We are admitting $+\infty$ as a value for $\|A^{-1}\|$.)

We shall use $\mathfrak{N}(A)$ to denote the null space of $A$, that is

$$\mathfrak{N}(A) = [x \mid Ax = 0].$$

Similarly $\mathfrak{N}(A^*)$ denotes the null space of $A^*$.

## 2. Preliminary lemmas.

LEMMA 1. *If $\mathfrak{X}_0$ is a finite-dimensional linear subspace of $\mathfrak{X}$, with basis $\{x_1, \cdots, x_n\}$, there exist $n$ elements $x_i^*$ of $\mathfrak{X}^*$ such that $x_i^*(x_j) = \delta_{ij}$, and $P(x) = \sum_{i=1}^n x_i^*(x)x_i$ is a projection of $\mathfrak{X}$ onto $\mathfrak{X}_0$. A similar result holds for $\mathfrak{X}_0^* \subset \mathfrak{X}^*$, with $Q(x^*) = \sum_{i=1}^n x^*(x_i)x_i^*$.*

The first part follows at once from the Hahn-Banach theorem, and the second part follows from more elementary considerations. Note that if the sets $\{x_1, \cdots, x_n\}$ and $\{x_1^*, \cdots, x_n^*\}$ are the same in the two cases, then $Q = P^*$.

LEMMA 2. *For each linear closed subspace $\mathfrak{X}_0$ of $\mathfrak{X}$, $OO\mathfrak{X}_0 = \mathfrak{X}_0$.*

This follows from the lemma in Banach [1, p. 57]. The corresponding result for subsets $\mathfrak{X}_0^*$ of $\mathfrak{X}^*$ does not hold in general, as Banach shows in [1, p. 115]. However, by an easy proof we have

LEMMA 3. *If $\mathfrak{X}_0^*$ is a finite-dimensional linear subspace of $\mathfrak{X}^*$, then $OO\mathfrak{X}_0^* = \mathfrak{X}_0^*$.*

LEMMA 4. *If $\{x_1^*, \cdots, x_n^*\}$ is a basis for $\mathfrak{X}_0^*$, $x_i^*(x_j) = \delta_{ij}$, and*

*$P(x)=\sum_{i=1}^{n} x_i^*(x)x_i$, then $\mathfrak{X}$ is the direct sum of $O\mathfrak{X}_0^*$ and the finite-dimensional space $P\mathfrak{X}$.*

To prove this, we observe that $x-P(x)$ is always in $O\mathfrak{X}_0^*$, and that if $x$ lies in $O\mathfrak{X}_0^*$, then $Px=0$.

LEMMA 5. *A linear transformation $A$ of $\mathfrak{X}$ into $\mathfrak{Y}$ is (completely) continuous if and only if its adjoint $A^*$ is (completely) continuous.*

**Proof.** This follows from Theorems 3 and 4 in Banach [1, p. 100]. To show that $A$ is completely continuous whenever $A^*$ is, we note that $A^{**}$, when restricted to $\mathfrak{X}$, reduces to $A$, so by Theorem 4 of Banach, $A$ transforms bounded sets in $\mathfrak{X}$ into sets in $\mathfrak{Y}$ which are compact in $\mathfrak{Y}^{**}$. But $\mathfrak{Y}$ is closed in $\mathfrak{Y}^{**}$.

LEMMA 6. *The range $A\mathfrak{X}$ of a linear continuous transformation $A$ of $\mathfrak{X}$ into $\mathfrak{Y}$ is closed if and only if $I(A)<\infty$.*

**Proof.** When $A\mathfrak{X}$ is closed it constitutes a Banach space, so the necessity of the condition follows from Banach [1, p. 38]. To see that the condition is sufficient, suppose $y_n=Ax_n$, and $\lim y_n=y$. By choosing a subsequence we may suppose that $\|y_{n+1}-y_n\|<1/2^n$. Then after $x_n$ has been suitably chosen, $x_{n+1}$ may be modified (still satisfying $y_{n+1}=Ax_{n+1}$) so that $\|x_{n+1}-x_n\| \leqq I(A)/2^n$. Hence the modified sequence $(x_n)$ has a limit $x$, and $Ax_n$ converges to $Ax=y$.

COROLLARY. *If $A\mathfrak{X}$ is closed, and* $\lim Ax_n=Ax$, *there exists a sequence $(x_n')$ such that $Ax_n'=Ax_n$ and* $\lim x_n'=x$.

**Proof.** From the definition of $I(A)$ we may choose $x_n'$ so that $A(x_n'-x)=A(x_n-x)$ and $\|x_n'-x\|\leqq(I(A)+1)\|A(x_n-x)\|$. (Compare Banach [1, p. 40, Theorem 4].)

LEMMA 7. *If $A$ is linear and continuous, $A\mathfrak{X}$ is closed if and only if $A^*\mathfrak{Y}^*$ is closed, and then $A^*\mathfrak{Y}^*=O\mathfrak{N}(A)$, $A\mathfrak{X}=O\mathfrak{N}(A^*)$, where $\mathfrak{N}(A)$ and $\mathfrak{N}(A^*)$ are the null spaces of $A$ and of $A^*$.*

For the proof, see Banach [1, pp. 149, 150, Theorems 8 and 9].

LEMMA 8. *If $A$ is linear and continuous, then $A$ maps $\mathfrak{X}$ onto the whole of the space $\mathfrak{Y}$ if and only if $A^*$ has a continuous inverse. Likewise $A$ has a continuous inverse if and only if $A^*$ maps $\mathfrak{Y}^*$ onto $\mathfrak{X}^*$.*

For the proof, see Banach [1, pp. 146–148, Theorems 1–4].

LEMMA 9. *If $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are two closed linear subspaces of $\mathfrak{X}$, $\mathfrak{X}_2$ being finite-dimensional, then $\mathfrak{X}_1+\mathfrak{X}_2$ (the linear space spanned by $\mathfrak{X}_1$ and $\mathfrak{X}_2$) is also closed.*

This is readily proved by induction on the dimension of $\mathfrak{X}_2$.

LEMMA 10. *Suppose* $\mathfrak{X}_1$ *and* $\mathfrak{X}_2$ *satisfy the conditions of* Lemma 9, *and also that* $\mathfrak{X}_1+\mathfrak{X}_2=\mathfrak{X}$. *Let* $E$ *be a linear continuous transformation of* $\mathfrak{X}$ *onto the whole of* $\mathfrak{Y}$. *Then* $E\mathfrak{X}_1$ *is closed.*

**Proof.** The space $\mathfrak{X}_3=E^{-1}E\mathfrak{X}_1$ equals $\mathfrak{X}_1+\mathfrak{X}_4$ where $\mathfrak{X}_4\subset\mathfrak{X}_2$, so $\mathfrak{X}_3$ is closed by Lemma 9. Let $E_3$ denote the restriction of $E$ to $\mathfrak{X}_3$. Then clearly $I(E_3)\leqq I(E)$. By Lemma 6, $I(E)<\infty$, and by the same Lemma 6, $E_3\mathfrak{X}_3=E\mathfrak{X}_1$ is closed.

3. **Proper values of a completely continuous transformation.** In this section we develop some properties of the family of transformations $T_c=E-cK$, described in §1. A complex number $c$ is called a *proper value* of $K$ in case the range of $T_c$ is a proper subset of $\mathfrak{Y}$, or equivalently, in case $T_c^*$ does not have an inverse.

In case $c=1$ is a proper value, the null space $\mathfrak{N}_1^*=\mathfrak{N}(T^*)$ is not null, and the range $\mathfrak{W}_1=T\mathfrak{X}\neq\mathfrak{Y}$. We set $\mathfrak{M}_1^*=E^*\mathfrak{N}_1^*$, and define $\mathfrak{Z}_1$ as the maximal subspace of $\mathfrak{X}$ satisfying $E\mathfrak{Z}_1=\mathfrak{W}_1$. This is the start of an iterative definition of four sequences of linear spaces $\mathfrak{N}_k^*$, $\mathfrak{M}_k^*$, $\mathfrak{W}_k$, $\mathfrak{Z}_k$, by means of the relations

$$T^*\mathfrak{N}_k^* \subset \mathfrak{M}_{k-1}^* \qquad (\mathfrak{N}_k^* \text{ maximal}),$$
$$\mathfrak{M}_k^* = E^*\mathfrak{N}_k^*,$$
$$\mathfrak{W}_k = T\mathfrak{Z}_{k-1},$$
$$E\mathfrak{Z}_k = \mathfrak{W}_k \qquad (\mathfrak{Z}_k \text{ maximal}).$$

It is obvious that

$$\mathfrak{N}_k^* \supset \mathfrak{N}_{k-1}^*, \qquad \mathfrak{M}_k^* \supset \mathfrak{M}_{k-1}^*,$$
$$\mathfrak{W}_k \subset \mathfrak{W}_{k-1}, \qquad \mathfrak{Z}_k \subset \mathfrak{Z}_{k-1}.$$

THEOREM 1. *The spaces* $\mathfrak{N}_k^*$ *and* $\mathfrak{M}_k^*$ *are all finite-dimensional, and* $\mathfrak{W}_k$ *and* $\mathfrak{Z}_k$ *are closed, and* $\mathfrak{W}_k=O\mathfrak{N}_k^*$, $\mathfrak{Z}_k=O\mathfrak{M}_k^*$.

**Proof.** Let $\mathfrak{B}$ be a bounded subset of $\mathfrak{N}_1^*$. Then $E^*\mathfrak{B}=K^*\mathfrak{B}$ is compact, since $K^*$ is completely continuous. Since $E^*$ has a bounded inverse, $\mathfrak{B}$ is compact. Thus $\mathfrak{N}_1^*$ is finite-dimensional, since every bounded subset is compact. (See Banach [1, p. 84, Theorem 8].) If $\mathfrak{M}_{k-1}^*$ is finite-dimensional, then $\mathfrak{N}_k^*$ is also. To show that $\mathfrak{W}_1=T\mathfrak{X}$ is closed, we shall show that $T^*\mathfrak{Y}^*$ is closed, and apply Lemma 7. Since $\mathfrak{N}_1^*$ is finite-dimensional, there is a projection $P$ of $\mathfrak{Y}^*$ onto $\mathfrak{N}_1^*$. If $T^*y_n^*$ tends to $x_0^*$, we may assume $Py_n^*=0$. If the sequence $(y_n^*)$ is bounded, we may (by choice of a subsequence) require that $K^*y_n^*$ converges, say to $x_1^*$. Then $E^*y_n^*$ tends to $x_0^*+x_1^*$ which must equal $E^*y_0^*$ for some $y_0^*$, since the range of $E^*$ is closed by Lemma 7. Then by Lemma 8, $y_n^*$ tends to $y_0^*$, so $K^*y_0^*=x_1^*$, and $T^*y_0^*=x_0^*$. The sequence $(y_n^*)$ must be bounded, since if $\|y_n^*\|$ tends to infinity, the sequence $y_{1n}^*=y_n^*/\|y_n^*\|$ satisfies the preceding conditions with $x_0^*=0$. Hence we would have $y_{1n}^*$ approaching $y_0^*$ with $\|y_0^*\|=1$, $P(y_0^*)=0$, $T^*y_0^*=0$, and so $y_0^*=P(y_0^*)$, which is a contradiction. Thus $T^*\mathfrak{Y}^*$ is closed, so $\mathfrak{W}_1 = T\mathfrak{X}$ is closed, and $\mathfrak{Z}_1 = E^{-1}\mathfrak{W}_1$ is also closed. Lemma 7 yields the further information that $\mathfrak{W}_1 = O\mathfrak{N}_1^*$, and it follows readily that $\mathfrak{Z}_1 = O\mathfrak{M}_1^*$.

Now suppose that $\mathfrak{Z}_k = O\mathfrak{M}_k^*$. Then $\mathfrak{Z}_k$ is closed, and since $\mathfrak{M}_k^*$ is finite-dimensional, $\mathfrak{X}$ is the direct sum of $\mathfrak{Z}_k$ and a finite dimensional space, by Lemma 4. Now $T$ maps $\mathfrak{X}$ onto the closed space $\mathfrak{W}_1$, so $\mathfrak{W}_{k+1} = T\mathfrak{Z}_k$ is closed, by Lemma 10 with $\mathfrak{X}_1$ replaced by $\mathfrak{Z}_k$, $E$ by $T$, and $\mathfrak{Y}$ by $\mathfrak{W}_1$. Then $\mathfrak{Z}_{k+1} = E^{-1}\mathfrak{W}_{k+1}$ is also closed and by a sequence of steps the relations $\mathfrak{N}_{k+1}^* = O\mathfrak{W}_{k+1}$, $\mathfrak{W}_{k+1} = O\mathfrak{N}_{k+1}^*$, $\mathfrak{Z}_{k+1} = O\mathfrak{M}_{k+1}^*$ are readily obtained. This completes the induction.

THEOREM 2. *There exists a least integer $\nu$ such that $\mathfrak{W}_{\nu+1} = \mathfrak{W}_\nu$, and then $\mathfrak{W}_{\nu+k} = \mathfrak{W}_\nu$ for all $k$.*

**Proof.** From the preceding theorem and the fact that $\mathfrak{N}_k^*$ and $\mathfrak{M}_k^*$ are in one-to-one correspondence, it is clear that $\mathfrak{W}_{\nu+1} = \mathfrak{W}_\nu$, $\mathfrak{Z}_{\nu+1} = \mathfrak{Z}_\nu$, $\mathfrak{N}_{\nu+1}^* = \mathfrak{N}_\nu^*$, $\mathfrak{M}_{\nu+1}^* = \mathfrak{M}_\nu^*$ all happen for the same index $\nu$. We make the proof by considering the nondecreasing sequence of finite-dimensional spaces $\mathfrak{M}_k^*$. If $\mathfrak{M}_{k+1}^* \neq \mathfrak{M}_k^*$ for every $k$, there exists a unit vector $x_{k+1}^* \in \mathfrak{M}_{k+1}^*$ whose distance from $\mathfrak{M}_k^*$ is unity. For each $k$ there is a point $y_k^* \in \mathfrak{N}_k^*$ with $E^*y_k^* = x_k^*$, $\|y_k^*\| \leq I(E^*)$. Then when $m > k$, $x_m^* - x_k^* = E^*y_m^* - E^*y_k^* = T^*y_m^* - T^*y_k^* + K^*y_m^* - K^*y_k^*$, and $T^*y_m^* \in \mathfrak{M}_{m-1}^*$, $T^*y_k^* \in \mathfrak{M}_{k-1}^* \subset \mathfrak{M}_{m-1}^*$, so $\|K^*y_m^* - K^*y_k^*\| \geq 1$ while $\|y_k^*\|$ is bounded, which contradicts the complete continuity of $K^*$.

We note that the space $\mathfrak{Z}_\nu$ is mapped onto $\mathfrak{W}_\nu$ by both $E$ and $T$. This pair of subspaces replaces the invariant subspace of the classical theory of Riesz.

We remark also that if we consider the sequence of spaces defined by the relations:

$$\mathfrak{N}_1 = \mathfrak{N}(T), \qquad \mathfrak{M}_k = E\mathfrak{N}_k,$$
$$T\mathfrak{N}_{k+1} \subset \mathfrak{M}_k, \qquad (\mathfrak{N}_{k+1} \text{ maximal}),$$

we may have $\mathfrak{N}_{k+1} \neq \mathfrak{N}_k$ for every $k$. For example, let $\mathfrak{X}$ and $\mathfrak{Y}$ be classical Hilbert space, and let the transformation $y = Ex$ be defined by $y_i = x_{i+3}$ while $y = Kx$ is defined by

$$y_1 = x_4, \qquad y_2 = x_5 - x_2, \qquad y_{i+2} = x_{i+4}/(i + 4).$$

Then a basis for $\mathfrak{N}_k$ is composed of $\delta_1, \delta_3, \delta_4, \alpha_1, \cdots, \alpha_k$, where $\delta_i$ has only its $i$th component different from zero, $\alpha_1$ has its $i$th component equal to $1/(i-1)!$ except the first four, which are zero, and $E\alpha_j = T\alpha_{j+1}$.

THEOREM 3. *Let $(c_j)$ be a sequence (finite or infinite) of distinct proper values of $K$. Let $\mathfrak{N}^{*(j)}$ denote the null space of $T_{c_j}^*$, and let $\mathfrak{W}^{(j)} = T_{c_j}\mathfrak{X}$. Let $[y_{ij}^* | i = 1, \cdots, k_j]$ be a basis for $\mathfrak{N}^{*(j)}$. Then the functionals $y_{ij}^*$ are finitely linearly independent, and there exists a system $[y_{ij}]$ of elements of $\mathfrak{Y}$ such that:*

(a) *for each $j$, $\mathfrak{W}^{(j)}$ and $[y_{ij} | i = 1, \cdots, k_j]$ span the space $\mathfrak{Y}$;*

(b) $y_{il}^*(y_{kj}) = \delta_{ik}\delta_{lj}$ *for* $l \leqq j$;
(c) $y_{kj}$ *lies in* $\mathfrak{W}^{(l)}$ *for* $l < j$;
(d) *the elements* $y_{ij}$ *are finitely linearly inuependent.*

**Proof.** Suppose

$$\sum_{j=1}^{n} \sum_{i=1}^{k_j} a_{ij} y_{ij}^* = 0.$$

Then

$$0 = \sum_{j=1}^{n-1} \sum_{i}^{*} (E - c_n K^*) a_{ij} y_{ij}^*$$

$$= E^* \sum_{j=1}^{n-1} \left(1 - \frac{c_n}{c_j}\right) \sum_{i} a_{ij} y_{ij}^*,$$

and since $E^*$ is one-to-one, this gives a relation involving only $n-1$ of the spaces $\mathfrak{N}^{*(i)}$. Hence by descent we arrive at a contradiction. By Theorem 1 for $k=1$, and Lemmas 1 and 4 we obtain (a) and (b) for $j=1$, $l=1$. Suppose that the system $[y_{ij}]$ has been determined so as to satisfy (a), (b), and (c) for $j<n$. Then the intersection $\mathfrak{V}^{(n-1)}$ of $\mathfrak{W}^{(1)}, \cdots, \mathfrak{W}^{(n-1)}$ is the orthogonal complement of the union of $\mathfrak{N}^{*(1)}, \cdots, \mathfrak{N}^{*(n-1)}$ and since the $y_{ij}^*$ are linearly independent, there are points $y_{in}$ in $\mathfrak{V}^{(n-1)}$ such that (b) holds for $l=j=n$. Property (a) holds for $j=n$, by the same argument as before. Property (d) follows at once from (b).

THEOREM 4. *The proper values of K have no finite accumulation point.*

**Proof.** Suppose $(c_n)$ is a bounded infinite sequence of distinct proper values. Let $\mathfrak{S}_n^*$ be the space spanned by $\mathfrak{N}^{*(1)}, \cdots, \mathfrak{N}^{*(n)}$. Then $(\mathfrak{S}_n^*)$ is a properly increasing sequence of finite-dimensional linear spaces, so each $\mathfrak{S}_n^*$ contains a unit vector $v_n^*$ at unit distance from $\mathfrak{S}_{n-1}^*$. The sequence $(c_n v_n^*)$ is bounded, so $(c_n K^* v_n^*)$ is compact. If $y^*$ is in $\mathfrak{N}^{*(i)}$, $T_c^* y^* = E^*(1 - c/c_j) y^*$, which is zero if $c = c_j$, so

$$T_{c_p}^* \mathfrak{S}_p^* \subset E^* \mathfrak{S}_{p-1}^*. \tag{1}$$

Now for $p>q$, let

$$d_{pq} = \left\| c_p K^* v_p^* - c_q K^* v_q^* \right\| = \left\| E^* v_p^* - T_{c_p}^* v_p^* - E^* v_q^* + T_{c_q}^* v_q^* \right\|,$$

so by (1)

$$d_{pq} = \left\| E^* v_p^* - E^* u^* \right\|, \quad \text{where} \quad u^* \text{ is in } \mathfrak{S}_{p-1}^* \quad \text{and} \quad \left\| v_p^* - u^* \right\| = 1.$$

Since the inverse of $E^*$ is bounded, by Lemma 8, this contradicts the compactness of $(c_n K^* v_n^*)$.

Since the range of $T_c^*$ may vary with $c$, we state properties of the inverse of $T_c^*$ as follows.

THEOREM 5. *If $c_0$ is not a proper value of $K$, and $(c_n)$ and $(y_n^*)$ are sequences such that* $\lim c_n = c_0$, $\lim T_{c_n}^* y_n^* = x_0^*$, *then $y_n^*$ tends to the unique solution $y_0^*$ of $T_{c_0}^* y^* = x_0^*$. Consequently if $T_c^* y^*(c) = x^*(c)$, and $x^*(c)$ is continuous (differentiable) at $c_0$, then $y^*(c)$ is continuous (differentiable) at $c_0$.*

**Proof.** Suppose first that $(y_n^*)$ is bounded. Then $\lim (T_{c_n}^* y_n^* - T_{c_0}^* y_n^*) = 0$ so $\lim T_{c_0}^* y_n^* = x_0^*$, and hence $\lim y_n^* = R x_0^*$, where $R$ is the inverse of $T_{c_0}^*$. If $\lim \|y_n^*\| = \infty$, set $v_n^* = y_n^* / \|y_n^*\|$. Then $\lim T_{c_n}^* v_n^* = 0$, so by the first case $\lim v_n^* = 0$, which is a contradiction.

With the help of the last theorem we see that a necessary and sufficient condition for $c_0$ to be a proper value of $K$ is that there exist sequences $(c_n)$ and $(y_n^*)$ such that:

(a) $\lim c_n = c_0$,

(b) $\|y_n^*\| = 1$,

(c) $\lim T_{c_n}^* y_n^* = 0$.

This suggests the following definitions:

(I) $c = \infty$ is a *singular value* for $K$ in case there exist sequences $(c_n)$ and $(y_n^*)$ such that (a'), (b), and (c) hold, where (a') means: $\lim c_n = \infty$.

(II) $c_0$ is a *proper value of order* $\mu$ for $K$ in case $\mu$ is the maximum order of vanishing of $T_c^* y^*(c)$, where $y^*(c)$ is a polynomial in $c$, not vanishing at $c_0$.

THEOREM 6. *If $c = 1$ is a proper value for $K$, it is a proper value of order $\nu$, where $\nu$ is the integer given by Theorem 2.*

**Proof.** It is readily verified by induction that $K^*$ (as well as $E^*$) maps $\mathfrak{N}_k^*$ onto $\mathfrak{M}_k^*$ one-to-one. Denoting by $D$ the operation of differentiation with respect to $c$, we have

$$D^k T_c^* y^*(c) = T_c^* D^k y^*(c) - k K^* D^{k-1} y^*(c).$$

Hence if $c = 1$ is a proper value of order $\mu$, there is a polynomial $y^*(c)$ such that $0 \neq y^*(1) \in \mathfrak{N}_1^*$, and by induction $D^k y^*(1) \in \mathfrak{N}_{k+1}^* - \mathfrak{N}_k^*$ for $k < \mu$. Hence $\mu \leqq \nu$. To show that $\mu = \nu$, we set $y^*(c) = \sum_{j=1}^{\nu} (c-1)^{j-1} \eta_j^*$, where $\eta_\nu^* \in \mathfrak{N}_\nu^* - \mathfrak{N}_{\nu-1}^*$ and $T_1^* \eta_j^* = K^* \eta_{j-1}^*$, so that no $\eta_j^* = 0$, and $T_c^* y^*(c) = -(c-1)^\nu K^* \eta_\nu^*$.

We next consider a decomposition of $K$ relative to a proper value, which we take for convenience to be $c = 1$. In view of the definition of the spaces $\mathfrak{N}_i^*$, there exists a basis $[\theta_{ij}^* | j = 1, \cdots, p_i; i = 1, \cdots, \nu]$ for $\mathfrak{N}_\nu^*$ such that $[\theta_{ij}^* | j = 1, \cdots, p_i; i = 1, \cdots, k]$ is a basis for $\mathfrak{N}_k^*$, and moreover

$$(2) \quad (E^* - K^*)\theta_{i+1,j}^* = E^* \theta_{ij}^* \quad \text{for} \quad j = 1, \cdots, p_{i+1}, \ i = 1, \cdots, \nu - 1.$$

Note that when the basis $[\theta_{ij}^*]$ is displayed by columns, the index $i$ ranges from 1 to $q_j$, and $j$ ranges from 1 to $p_1$. Choose $\eta_{ij}$ from $\mathfrak{Y}$ so that

$$\theta^*_{ij}(\eta_{kl}) = \delta_{ik}\delta_{jl}, \tag{3}$$

and set

$$Q_j(y) = \sum_{i=1}^{q_j} \theta^*_{ij}(y)\eta_{ij}, \qquad Q = \sum_{j=1}^{p_1} Q_j,$$

$$Q^*_j(y^*) = \sum_{i=1}^{q_j} y^*(\eta_{ij})\theta^*_{ij},$$

$$G_j = Q_jK, \qquad G = \sum_{j=1}^{p_1} G_j, \qquad H = K - G. \tag{4}$$

Then clearly $Q^*$ projects $\mathfrak{Y}^*$ onto $\mathfrak{N}^*_\nu$, and $Q^*_j$ projects $\mathfrak{Y}^*$ onto the subspace of $\mathfrak{N}^*_\nu$ corresponding to a single proper vector $\theta^*_{1j}$ by means of the relations (2).

THEOREM 7. *The decomposition of* $K$ *given in* (4) *has the following properties*:

(a) *each* $G_j$ *has only one independent proper vector* $\theta^*_{1j}$, *which corresponds to the proper value* $c=1$;

(b) $G$ *has only the proper value* $c=1$, *and the null-spaces* $\mathfrak{N}^*_k$ *for* $G$ *are the same as those for* $K$;

(c) $c=\infty$ *is not a singular value for* $G$;

(d) $c=1$ *is not a proper value for* $H$;

(e) $G^*_l y^*=0$ *for every* $y^*$ *satisfying* $E^*y^*=G^*_j y_1^*$ *for some* $y_1^*$ *and some* $j\neq l$;

(f) $H^*y^*=0$ *for every* $y^*$ *satisfying* $E^*y^*=G^*y_1^*$ *for some* $y_1^*$;

(g) *if* $c$ *is a proper value for* $K$ *and* $c\neq 1$, *then* $c$ *is a proper value for* $H$, *and conversely.*

**Proof.** If $(E^*-cG^*_j)y^*=0$, it follows from (2) and the fact that $E^*$ is one-to-one that $y^*$ is a linear combination of $\theta^*_{1j}, \cdots, \theta^*_{q_j,j}$, and by another application of (2) that $c=1$ and $y^*$ is a multiple of $\theta^*_{1j}$. This proves (a). To prove (b), we observe that $Q^*$ projects $\mathfrak{Y}^*$ onto $\mathfrak{N}^*_\nu$, and from (2) that $K^*$ maps $\mathfrak{N}^*_\nu$ onto $\mathfrak{M}^*_\nu$ one-to-one, so $G^*$ maps $\mathfrak{Y}^*$ onto $\mathfrak{M}^*_\nu$, and $G^*=K^*$ on $\mathfrak{N}^*_\nu$. Thus if $(E^*-cG^*)y^*=0$, $y^*$ is in $\mathfrak{N}^*_\nu$, and $(c-1)E^*y^*=c(E^*-G^*)y^*=c(E^*-K^*)y^*$, so if $c\neq 1$, $y^*$ is in $\mathfrak{N}^*_{\nu-1}$. Proceeding by descent, we finally find $y^*=0$. The remainder of (b) follows readily. If we suppose that $c=\infty$ is a singular value for $G$, there must exist sequences $(c_n)$ and $(y_n^*)$ with $\lim c_n=\infty$, $\|y_n^*\|=1$, $\lim (E^*-c_nG^*)y_n^*=0$. Since the sequence $(E^*y_n^*)$ is bounded, $(c_nG^*y_n^*)$ is likewise bounded and lies in the finite-dimensional space $\mathfrak{M}^*_\nu$, so we may suppose that it converges to a point $x^*$ in $\mathfrak{M}^*_\nu$. Then $x^*=E^*y^*$ where $y^*$ is in $\mathfrak{N}^*_\nu$, and since $E^*$ has a continuous inverse, $\lim y_n^*=y^*$, so $\|y^*\|=1$. But $\lim G^*y_n^*=0=G^*y^*$, and since $G^*$ gives a one-to-one correspondence between $\mathfrak{N}^*_\nu$ and $\mathfrak{M}^*_\nu$, we have $y^*=0$, which is a contradiction. This proves (c). To secure (d) we observe that if $0=(E^*-H^*)y^*=(E^*-K^*)y^*+G^*y^*$, then $y^*$ is in $\mathfrak{N}^*_\nu$ since $G^*y^*$ is in $\mathfrak{M}^*_\nu$, so $H^*y^*=0$, $E^*y^*=0$,

$y^*=0$. We next prove (e) and (f). If $E^*y^*=G_j^*y_1^*$, we have from (2) (setting $\theta_{0j}^*=0$)

$$y^* = \sum_{i=1}^{q_j} y_1^*(\eta_{ij})(\theta_{ij}^* - \theta_{i-1,j}^*), \tag{5}$$

since $E^*$ is one-to-one, and

$$G_j^* y_1^* = \sum_{i=1}^{q_j} y_1^*(\eta_{ij}) K^* \theta_{ij}^*. \tag{6}$$

So by (5) and (6) and (3),

$$G^* y^* = \sum_{i=1}^{q_j} \sum_{m=1}^{q_l} y_1^*(\eta_{ij}) [\theta_{ij}^*(\eta_{ml}) - \theta_{i-1,j}^*(\eta_{ml})] K^* \theta_{ml}^* = \delta_{jl} K^* y^*,$$

and hence $H^*y^*=0$. Property (f) follows immediately. Finally, to prove (g), we observe that if $c\neq 1$, $y^*\neq 0$, and $(E^*-cK^*)y^*=0$, $G^*y^*=E^*y_1^*$, where $y_1^*$ is in $\mathfrak{N}_\nu^*$, and so $H^*y_1^*=0$, by (f). Hence

$$\begin{aligned} 0 &= E^*y^* - cG^*y^* - cH^*y^* = E^*y^* - cE^*y_1^* - cH^*y^* + c^2H^*y_1^* \\ &= (E^* - cH^*)(y^* - cy_1^*). \end{aligned}$$

Now if $y^*=cy_1^*$ we would have $0=(E^*-cK^*)y_1^*=(E^*-cG^*)y_1^*$, and this cannot happen for $c\neq 1$ by part (b) of the theorem. So $c$ is a proper value of $H$. For the converse, let $(E^*-cH^*)y^*=0$ with $y^*\neq 0$. Then $y^*$ is not in $\mathfrak{N}_\nu^*$, and there is a finite sequence $y_1^*, \cdots, y_k^*$, such that $G^*y^*=E^*y_1^*$, $(E^*-K^*)y_i^*=E^*y_{i+1}^*$ for $i=1, \cdots, k-1$, each $y_i^*$ is in $\mathfrak{N}_\nu^*$, $y_k^*$ is in $\mathfrak{N}_1^*$, and so

$$(E^* - cK^*)\left(y^* + \sum_{i=1}^{k} \alpha_i y_i^*\right) = E^*\left[- cy_1^* + (1 - c)\sum_{i=1}^{k} \alpha_i y_i^* + c\sum_{i=1}^{k-1} \alpha_i y_{i+1}^*\right].$$

The expression in square brackets vanishes if $(1-c)\alpha_1=c$, $(1-c)\alpha_i=-c\alpha_{i-1}$ for $i=2, \cdots, k$.

## Bibliography

1. S. Banach, *Théorie des opérations linéaires*, 1932.

2. F. Riesz, *Über lineare Funktionalgleichungen*, Acta Math. vol. 41 (1918) pp. 71–98.

3. T. H. Hildebrandt, *Über vollstetige lineare Transformationen*, Acta Math. vol. 51 (1928) pp. 311–318.

4. J. Schauder, *Über lineare vollstetige Funktionaloperationen*, Studia Mathematica vol. 2 (1930) pp. 183–196.

5. L. Schwartz, *Homomorphismes et applications complètement continues*, C. R. Acad. Sci. Paris vol. 236 (1953) pp. 2472–2473. This note gives an extension to linear topological spaces of part of Theorem 1 above.

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